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MHV vertices and tree amplitudes in gauge theory

Freddy Cachazo,[a] Peter Svrcek[b] and Edward Witten[a]

[a] *School of Natural Sciences, Institute for Advanced Study, Princeton NJ 08540 U.S.A.*
[b] *Joseph Henry Laboratories, Princeton University Princeton NJ 08544 U.S.A.*
E-mail: cachazo@ias.edu, psvrcek@princeton.edu, witten@ias.edu

ABSTRACT: As an alternative to the usual Feynman graphs, tree amplitudes in Yang-Mills theory can be constructed from tree graphs in which the vertices are tree level MHV scattering amplitudes, continued off shell in a particular fashion. The formalism leads to new and relatively simple formulas for many amplitudes, and can be heuristically derived from twistor space.

KEYWORDS: Duality in Gauge Field Theories, QCD.

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Contents

1. Introduction

Perturbative scattering amplitudes in Yang-Mills theory have remarkable properties that are not apparent from the textbook recipes for computing them. Unexpected selection rules for helicity amplitudes were uncovered in the earliest computation of tree level gluon scattering [2]. Tree amplitudes in which the maximal number of gluons have the same helicity are described by a marvelously simple formula [1, 3]. (These are known as maximal helicity violating or MHV amplitudes.) Loop amplitudes also turn out to be unexpectedly simple [4, 5].

Some properties of perturbative Yang-Mills theory may apparently be explained [6] by relating this theory to the instanton expansion of a certain string theory in twistor space [7]. In the present paper, we reconsider the tree amplitudes of perturbative Yang-Mills theory in a way that is suggested by the twistor transform (and by our study of differential equations obeyed by scattering amplitudes, which will appear elsewhere) and also by the use [4, 5] of MHV tree amplitudes in calculating loop amplitudes.

Consider a theory in Minkowski space of gauge invariant local fields such as scalar fields ϕ_i. We consider a local interaction vertex such as a polynomial interaction $W = \int d^4x\, F(\phi_i)$. A point in Minkowski space corresponds [7] to a "line" in twistor space — that is, to a linearly embedded copy of $\mathbb{CP}^1$. So the interaction vertex $F(\phi_i)$, which is supported on a point in Minkowski space, is supported on a line in twistor space.

As shown in [6], the tree level MHV amplitudes for scattering of any number of gluons of positive helicity and two of negative helicity is similarly supported on a line. So we think of this amplitude as representing, in some sense, a generalization of a local interaction vertex.

JHEP09(2004)006

This is in the spirit of analyses of loop diagrams [4, 5] in which, roughly speaking, MHV tree level amplitudes are regarded as interactions and amplitudes that are rational functions (of the spinor variables used to describe external particles) are considered to be "local."

In this paper, we pick a specific off-shell continuation of the MHV amplitude and consider Feynman diagrams in which the vertices are tree level MHV amplitudes — with an arbitrary number of gluon lines — and the propagator is the standard Feynman propagator $1/p^2$. We call these diagrams MHV diagrams.

Our off-shell continuation of the MHV amplitude is not Lorentz-covariant, and the sum of MHV diagrams is not manifestly Lorentz-covariant. Nevertheless, we argue that the sum of MHV tree diagrams is covariant, and we verify, for examples with five, six, or seven gluons, that this sum coincides with conventional Yang-Mills tree amplitudes.

Assuming that this is so to all orders, we obtain relatively short and simple expressions for certain amplitudes, such as the helicity amplitudes $---+++\ldots+$. See [8] for previously known formulas for these amplitudes.

In section 2, we describe our off-shell continuation. In section 3, we describe explicit computations of some amplitudes. In section 4, we verify that MHV tree amplitudes have the same collinear and multiparticle singularities as the standard Yang-Mills tree amplitudes. In section 5, we prove that the sum of MHV tree amplitudes is Lorentz-covariant. Finally, in section 6, we attempt to justify the MHV tree amplitudes as a method of evaluating the twistor amplitudes coming from completely disconnected instantons (that is, from collections of disjoint instantons each of which has instanton number one). This argument is not really rigorous as the rules for what integration contours to use in twistor space are not entirely clear.

The argument in section 6 raises a puzzle to which we do not have an answer. Other recent results suggest that it is possible to compute the same amplitudes solely from connected instantons [9]. Why might it be possible to compute the same amplitudes from connected instantons or from completely disconnected ones? Perhaps in some topological string theory, it is possible to choose one integration contour in field space that picks up only the connected instantons and another one that picks up only the completely disconnected instantons.

2. Definition of MHV tree amplitudes

We recall that in four dimensions, a momentum vector p_μ can conveniently be represented as a bispinor $p_{a\dot{a}}$ and that the momentum vector for a massless particle can be factored as $p_{a\dot{a}} = \lambda_a\widetilde{\lambda}_{\dot{a}}$ in terms of spinors λ_a, $\widetilde{\lambda}_{\dot{a}}$ of positive and negative chirality. Spinor inner products are denoted as $\langle\lambda, \lambda'\rangle = \epsilon_{ab}\lambda^a\lambda'^b$, $[\widetilde{\lambda}, \widetilde{\lambda}'] = \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}^{\dot{a}}\widetilde{\lambda}'^{\dot{b}}$. If $p_{a\dot{a}} = \lambda_a\widetilde{\lambda}_{\dot{a}}$, $q_{a\dot{a}} = \lambda'_a\widetilde{\lambda}'_{\dot{a}}$, then $2p\cdot q = \langle\lambda, \lambda'\rangle[\widetilde{\lambda}, \widetilde{\lambda}']$. For more detail and references, see [6].

We will be studying tree level scattering amplitudes with n gluons. Such an amplitude is in a natural way a sum of subamplitudes associated with different cyclic orderings of the external gluons; we focus on the term associated with a particular cyclic ordering, say the one for which the group theory factor is Tr $T_1T_2\ldots T_n$. We suppress this factor in writing the amplitudes.

A tree level scattering amplitude with n gluons more than $n-2$ of which have the same helicity vanishes. The amplitudes with $n-2$ gluons all of the same helicity are called maximally helicity violating or MHV amplitudes. The MHV tree amplitude with $n-2$ gluons of positive helicity are as follows [1, 3]. If the gluons of negative helicity are labeled x, y (which may be any integers from 1 to n), the amplitude is

$$A_n = \frac{\langle \lambda_x, \lambda_y \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle}. \tag{2.1}$$

(We omit the trace Tr $T_1 \dots T_n$, a delta function $(2\pi)^4 \delta^4(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}})$ of energy-momentum conservation, and a factor g^{n-2}, with g the Yang-Mills coupling.)

In this paper, we will continue these "mostly plus" MHV amplitudes off-shell and use them as vertices in tree diagrams that we will call MHV tree diagrams.[1] (We do not include additional vertices for the "mostly minus" MHV tree amplitudes; along with other amplitudes, they are computed from trees with "mostly plus" vertices.) In the physical amplitude (2.1), each particle is assumed to be on-shell, with lightlike momentum vector $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$. To generalize the MHV tree amplitude to a vertex that can be inserted in a Feynman diagram, we need to continue it off-shell. An off-shell field is still characterized by a momentum vector $p_{a\dot{a}}$, but what can be meant by λ_a if p is not lightlike?

Suppose that $p_{a\dot{a}}$ is lightlike. We can pick an arbitrary negative chirality spinor $\eta^{\dot{a}}$ and then up to scaling we can take $\lambda_a = p_{a\dot{a}} \eta^{\dot{a}}$. In fact, if $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$, then $\lambda_a = p_{a\dot{a}} \eta^{\dot{a}} / [\widetilde{\lambda}, \eta]$. The factor $1/[\widetilde{\lambda}, \eta]$ is irrelevant since tree amplitudes that we compute will always be invariant under rescaling of the λ's for all the off-shell, internal lines.

This leads to our definition of the off-shell continuation. We simply pick an arbitrary $\eta^{\dot{a}}$ and then define λ_a for any internal line carrying momentum $p_{a\dot{a}}$ in a Feynman diagram by

$$\lambda_a = p_{a\dot{a}} \eta^{\dot{a}} \,. \tag{2.2}$$

For example, if $\eta^{\dot{a}} = \delta^{\dot{a}2}$, then the definition is $\lambda_a = p_{a\dot{2}}$. We use the same η for all the off-shell lines in all diagrams contributing to a given amplitude. For external lines — lines representing incoming or outgoing gluons in a scattering process – λ is defined in the usual way in terms of the wave function of the initial or final particle. With this understanding of what λ means for each particle, we simply take the "mostly plus" MHV scattering amplitude (2.1) as the n-gluon vertex in our Feynman diagram, for all $n \geq 3$. (We introduce no additional vertices for the "mostly minus" MHV amplitudes. They will be computed from tree diagrams using mostly plus vertices, as we see in the next section.)

At each interaction vertex, each gluon, understood to be incoming, is assigned a definite helicity. This is so for both on-shell and off-shell lines. In fact, at an n-gluon vertex, $n-2$ of the gluons have positive helicity and two have negative helicity; in (2.1), the two gluons of negative helicity have been labeled x, y. If a gluon is considered to be outgoing, its helicity label is reversed.

[1] A "mostly plus" MHV amplitude has two gluons of negative helicity and any number of positive helicity. In the exceptional cases that the number of positive helicity gluons is one or two, there is not really a majority of gluons with positive helicity.

JHEP09(2004)006

For the propagator of an off-shell gluon of momentum p, we take simply $1/p^2$. The two ends of any propagator must have opposite helicity labels — plus at one end and minus at the other end — because an incoming gluon of one helicity is equivalent to an outgoing gluon of the opposite helicity.

In the next section, we give some examples of Feynman diagrams computed using these rules. For now, we simply make the following observation. Consider a tree diagram with v vertices and (therefore) $v-1$ propagators connecting them. (See figure 1 for an example.) Each vertex has precisely two gluon lines of negative helicity emanating from it. So a total of $2v$ negative helicity gluon lines emanate from the vertices. Each of the $v-1$ propagators connects at precisely one end to one of these $2v$ lines. This leaves $v+1$ negative helicity lines that must be attached to external particles. In other words, a tree level scattering amplitude with q external gluons of negative helicity must be obtained from an MHV tree diagram with v vertices such that $q = v+1$ or equivalently

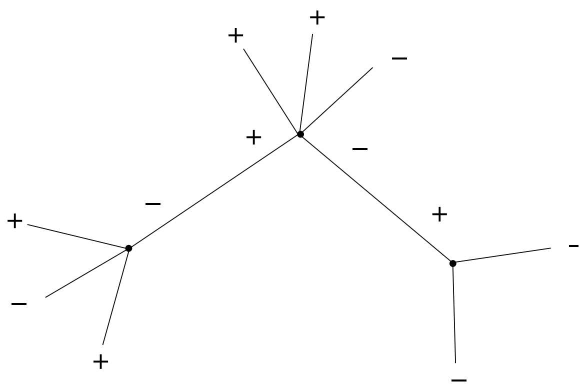


Figure 1: A tree diagram with MHV vertices. In this example, the number of vertices is $v = 3$; they are connected by $v - 1 = 2$ propagators. The vertices are respectively trivalent, four-valent, and five-valent. Internal and external lines are labeled by their helicity.

$$v = q - 1\,. \tag{2.3}$$

This implies, in particular, that MHV tree diagrams with $q < 2$ external gluons vanish, since they contain no vertices at all. This is in agreement with the fact that these amplitudes vanish in Yang-Mills theory. Moreover, if $q = 2$, the number of vertices is $v = 1$, and the MHV tree amplitude is equal by definition to the Yang-Mills tree amplitude. The first nontrivial case of our claim is for $q = 3$, $v = 2$.

The result (2.3) is analogous to the result in [6] that a Yang-Mills tree amplitude with q gluons of negative helicity (and any number of positive helicity) must be derived from a curve in twistor space of degree or instanton number $d = q - 1$. The degree one curves in twistor space correspond to the MHV vertices in the present approach.

3. Examples

Here we will describe examples of evaluation of MHV tree amplitudes. As just explained, the first case to consider is that the number of negative helicity gluons is $q = 3$ and the number of vertices is therefore $v = 2$.

We begin with the first case, the four gluon amplitude with momenta $p_1, \ldots, p_4$ and helicities $+ - - -$. This vanishes in Yang-Mills theory; we want to verify that it also vanishes when computed from MHV tree diagrams. As indicated in figure 2, there are two diagrams to consider. In the first diagram, there is an internal line with momentum

JHEP09(2004)006

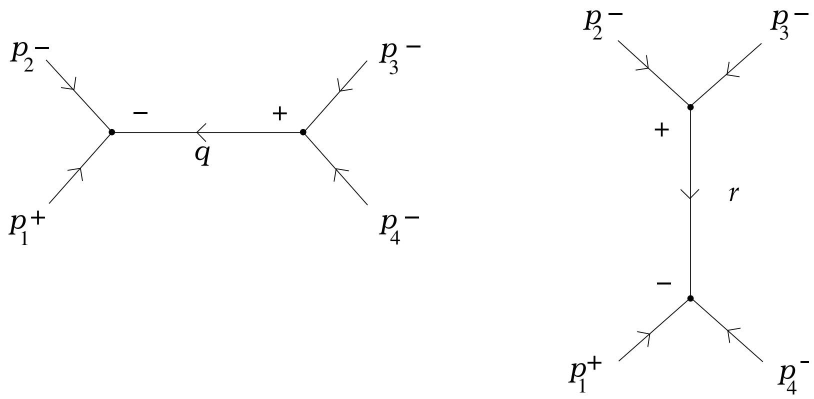


Figure 2: MHV tree diagrams contributing to the $+---$ amplitude, which is expected to vanish. Arrows indicate the momentum flow, while $+$ and $-$ signs denote the helicity.

$q = -p_1 - p_2 = p_3 + p_4$. We write λ_q, $\widetilde{\lambda}_q$ for the corresponding spinors. As explained in section 2, $\lambda_{q\,a} = q_{a\dot{a}}\eta^{\dot{a}}$, for some arbitrary $\eta^{\dot{a}}$ (which we take to be the same in both diagrams). We abbreviate $\widetilde{\lambda}_{i\,\dot{a}}\eta^{\dot{a}}$ as ϕ_i. So

$$\lambda_{q\,a} = -\lambda_{1\,a}\phi_1 - \lambda_{2\,a}\phi_2 = \lambda_{3\,a}\phi_3 + \lambda_{4\,a}\phi_4\,, \tag{3.1}$$

where we have used the fact that $p_{i\,a\dot{a}} = \lambda_{i\,a}\widetilde{\lambda}_{i\,\dot{a}}$. The amplitude associated with the first diagram in figure 2 is

$$\frac{\langle\lambda_2,\lambda_q\rangle^3}{\langle\lambda_q,\lambda_1\rangle\langle\lambda_1,\lambda_2\rangle}\frac{1}{q^2}\frac{\langle\lambda_3,\lambda_4\rangle^3}{\langle\lambda_4,\lambda_q\rangle\langle\lambda_q,\lambda_3\rangle}\,. \tag{3.2}$$

To obtain this formula, we took the propagator to be $1/q^2$, and we read off the trivalent vertices from (2.1).

From (3.1), we have $\langle\lambda_2,\lambda_q\rangle = -\langle 2\ 1\rangle\phi_1$, $\langle\lambda_q,\lambda_1\rangle = -\langle 2\ 1\rangle\phi_2$, $\langle\lambda_4,\lambda_q\rangle = \langle 4\ 3\rangle\phi_3$, and $\langle\lambda_q,\lambda_3\rangle = \langle 4\ 3\rangle\phi_4$. (We recall that $\langle i\ j\rangle$ is an abbreviation for $\langle\lambda_i,\lambda_j\rangle$.) So (3.2) becomes

$$\frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 2\ 1\rangle^3}{\langle 2\ 1\rangle\langle 1\ 2\rangle}\frac{1}{q^2}\frac{\langle 3\ 4\rangle^3}{\langle 4\ 3\rangle\langle 4\ 3\rangle}\,. \tag{3.3}$$

Using $q^2 = (p_1+p_2)^2 = 2p_1\cdot p_2 = \langle 1\ 2\rangle[1\ 2]$, and $\langle i\ j\rangle = -\langle j\ i\rangle$, this becomes

$$-\frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 3\ 4\rangle}{[2\ 1]}\,. \tag{3.4}$$

A very similar evaluation of the second diagram gives

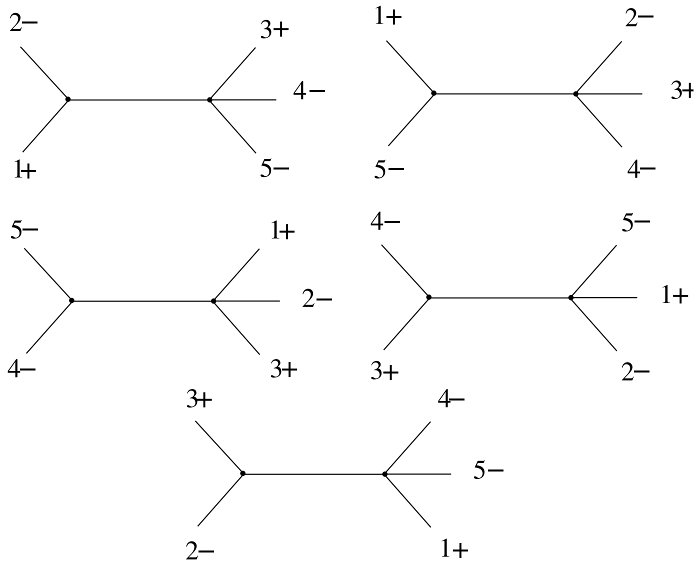
$$-\frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 3\ 2\rangle}{[4\ 1]}\,. \tag{3.5}$$

In fact, a new evaluation is not needed, since the second diagram can be obtained from the first by exchanging particles 2 and 4. The sum of these expressions vanishes, since momentum conservation implies that $0 = \sum_i\langle 3\ i\rangle[i\ 1] = \langle 3\ 2\rangle[2\ 1] + \langle 3\ 4\rangle[4\ 1]$.

The next case is the five gluon amplitude with two gluons of positive helicity and three of negative helicity. In general, five MHV tree diagrams contribute, as sketched in

JHEP09(2004)006

(a)



(b)

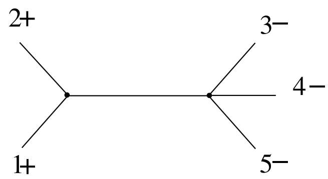


Figure 3: (a) MHV tree diagrams contributing to the $+-+--$ amplitude. (b) This contribution to the $++---$ amplitude is absent, as there is no $++-$ vertex.

figure 3a for the case of helicities $+-+--$. Each diagram contains two vertices, one of them trivalent and one four-valent. The vertices are all defined off-shell by the same procedure as above. The sum of the five MHV tree diagrams can be shown, with the aid of symbolic manipulation, to coincide with the standard tree level amplitude for this process, which is

$$\frac{[1\ 3]^4}{[1\ 2][2\ 3][3\ 4][4\ 5][5\ 1]}\,. \tag{3.6}$$

For the helicity configuration $++---$, there are only four MHV tree diagrams; there is no contribution of the form sketched in figure 3b, since by definition each vertex in an MHV tree diagram absorbs precisely two gluon lines of negative helicity. We have verified with the help of symbolic manipulation that the sum of the four remaining diagrams reproduces the standard result (which is obtained from (3.6) by simply replacing $[1\ 3]^4$ in the numerator by $[1\ 2]^4$).

We have made similar verifications in a number of additional cases, including five gluon amplitudes with helicity $+----$, all the six gluon amplitudes with three or four gluons of negative helicity, all the seven gluon amplitudes with three gluons of negative helicity, and finally the $++-----$ amplitude.

JHEP09(2004)006

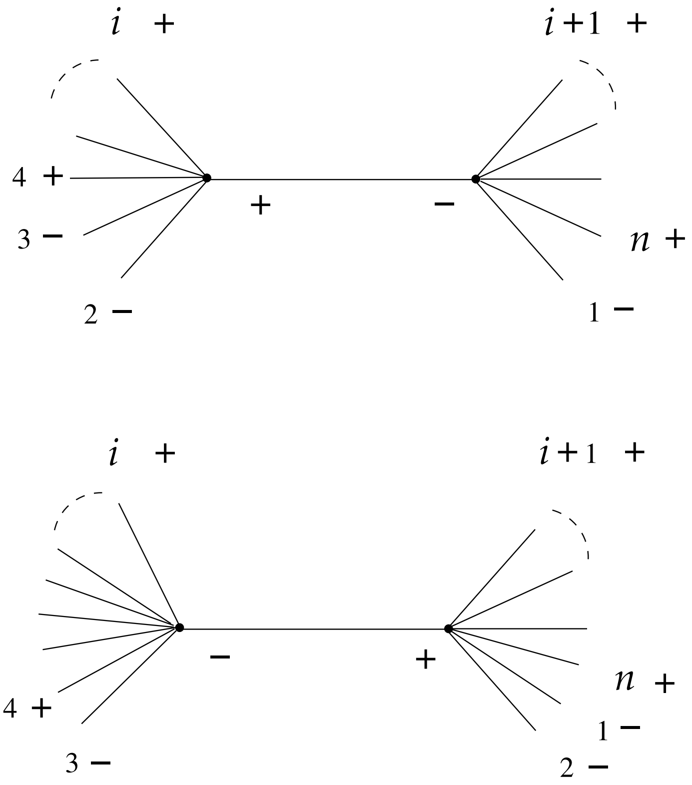


Figure 4: MHV tree diagrams contributing to the $---++\ldots+$ amplitude.

Our claim that conventional Yang-Mills tree amplitudes coincide with the amplitudes computed from MHV tree diagrams implies surprisingly simple formulas for some amplitudes. For example, consider n gluon amplitudes with precisely three gluons of negative helicity — the next case after the simple MHV amplitudes. They come from MHV tree diagrams with precisely two vertices and one propagator. If the helicities are $---+++\ldots++$, i.e., the three gluons of negative helicity are consecutive (see [8] for a previous evaluation of these amplitudes), there are precisely $2(n-3)$ possible diagrams, sketched in figure 4. They can be evaluated to give

$$\begin{aligned} A = & \sum_{i=3}^{n-1} \frac{\langle 1\ (2,i)\rangle^3}{\langle (2,i)\ i+1\rangle \langle i+1\ i+2\rangle \cdots \langle n\ 1\rangle} \frac{1}{q_{2i}^2} \frac{\langle 2\ 3\rangle^3}{\langle (2,i)\ 2\rangle \langle 3\ 4\rangle \cdots \langle i\ (2,i)\rangle} + \\ & + \sum_{i=4}^{n} \frac{\langle 1\ 2\rangle^3}{\langle 2\ (3,i)\rangle \langle (3,i)\ i+1\rangle \cdots \langle n\ 1\rangle} \frac{1}{q_{3i}^2} \frac{\langle (3,i)\ 3\rangle^3}{\langle 3\ 4\rangle \cdots \langle i-1\ i\rangle \langle i\ (3,i)\rangle} . \end{aligned} \tag{3.7}$$

Here we define $q_{ij} = p_i + p_{i+1} + \cdots + p_j$; the corresponding spinor $\lambda_{ij\,a}$ is defined in the usual way as $\lambda_{ij\,a} = q_{ij\,a\dot{a}}\eta^{\dot{a}}$, and $\langle i\ (j,k)\rangle$ is an abbreviation for $\langle \lambda_i, \lambda_{jk}\rangle$.

For other orderings of the external helicities, the number of diagrams is greater, but grows for large n at most as n^2. If S, T, and U are the number of positive helicity gluons between successive gluons of negative helicity (so $S+T+U = n-3$), then the number of diagrams is $2(n-3) + ST + TU + US$.

JHEP09(2004)006

The formula (3.7) is not manifestly covariant in general, but it becomes so if we pick $\eta^{\dot a}$ to equal one of the $\widetilde{\lambda}_i^{\dot a}$. (We show in section 5 that the amplitude is independent of the choice of $\eta^{\dot a}$.) If $\eta^{\dot a} = \widetilde{\lambda}_2^{\dot a}$, the amplitude becomes

$$A = \frac{1}{\prod_{k=3}^{n}\langle k\ k+1\rangle}\left[\sum_{i=4}^{n-1}\frac{\langle i\ i+1\rangle}{\langle i^-|\not{q}_{2,i}|2^-\rangle\langle (i+1)^-|\not{q}_{i+1,2}|2^-\rangle\langle 2^-|\not{q}_{2,i}|2^-\rangle}\times\right.$$
$$\left.\times\left(\frac{\langle 3\ 2\rangle^3\langle 1^-|\not{q}_{2,i}|2^-\rangle^3}{q_{2,i}^2}+\frac{\langle 1\ 2\rangle^3\langle 3^-|\not{q}_{i+1,2}|2^-\rangle^3}{q_{i+1,2}^2}\right)+A_{3,n}\right] \tag{3.8}$$

where we have introduced the manifestly Lorentz covariant notation $\langle m^-|\not{p}\ |r^-\rangle = m^a p_{a\dot a} r^{\dot a}$ and used the fact that $q_{3,i} = -q_{i+1,2}$. $A_{3,n}$ is the contribution from the $i = 3$ and $i = n$ terms of the first and second sums in (3.7) respectively. We have to treat these terms separately, because they have a factor of $[2\ \eta]$ in the denominator, which vanishes for $\eta^{\dot a} = \widetilde{\lambda}_2^{\dot a}$. However, combining them and using Schouten's identity[2] one finds a factor of $[2\ \eta]$ in the numerator as well. Thus, the substitution $\eta^{\dot a} = \widetilde{\lambda}_2^{\dot a}$ can be made to get

$$A_{3,n} = -\langle 1\ 3\rangle^2\left(\frac{s_{13}+2(s_{12}+s_{23})}{[3\ 2][1\ 2]}+\frac{\langle 1\ 2\rangle\langle n\ 3\rangle}{[1\ 2]\langle n\ 1\rangle}+\frac{\langle 3\ 2\rangle\langle 1\ 4\rangle}{[3\ 2]\langle 3\ 4\rangle}\right) \tag{3.9}$$

where $s_{km} = (p_k + p_m)^2 = \langle k\ m\rangle[k\ m]$.

The amplitude (3.8) is manifestly Lorentz-covariant and bose-symmetric. Bose symmetry merely says that the amplitude should be invariant under rotations and reflections of the chain that preserve the helicities. For this particular amplitude, the only such symmetry is the reflection that maps particle k to particle $4-k$; we have chosen η in a way that preserves this symmetry. One could also obtain different but manifestly Lorentz-covariant and bose-symmetric expressions for the same amplitude by averaging over the choices $\eta^{\dot a} = \widetilde{\lambda}_k^{\dot a}$ and $\eta^{\dot a} = \widetilde{\lambda}_{4-k}^{\dot a}$, for some fixed k. (Because of Schouten's identity, the various spinor products are not independent, and quite different-looking formulas can be written for the same amplitudes.) We have verified that (3.8) agrees with the standard result for the $---+++$ amplitude.

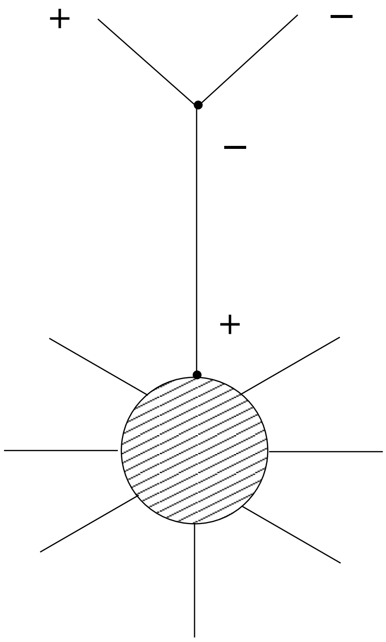


Figure 5: Diagrams contributing to collinear singularities of type (b). The shaded "blob" represents the complete $n-1$ gluon tree amplitude.

4. Collinear and multiparticle singularities

Here we will show that MHV tree graphs generate amplitudes with the correct collinear and multiparticle singularities.

Collinear singularities arise when (for example) the momenta of two incoming particles in a scattering amplitude are proportional, so that their sum is also lightlike.

[2]This identity asserts that for any four spinors $\alpha,\beta,\gamma,\delta$, we have $\langle\alpha,\beta\rangle\langle\gamma,\delta\rangle+\langle\alpha,\gamma\rangle\langle\delta,\beta\rangle+\langle\alpha,\delta\rangle\langle\beta,\gamma\rangle=0$.

JHEP09(2004)006

We describe a collinear singularity as a process with two particles going to one, so we consider, for example, the collinear singularity $++ \to +$ with two initial gluons of positive helicity combining to one of positive helicity. Since crossing symmetry reverses the helicity of a gluon, and our convention for vertices in a Feynman diagram is to consider all gluons incoming, the $++ \to +$ collinear singularity receives a contribution from a $++-$ interaction vertex.

In MHV tree diagrams, a singularity when two gluons i and $i+1$ develop collinear momenta will only arise if these two gluons are attached to the same vertex in the graph. In Yang-Mills theory, the collinear singularities are $++ \to +$, $+- \to -$, $+- \to +$, and $-- \to -$. (There are no $++ \to -$ or $-- \to +$ collinear singularities, since there are no $+++$ or $---$ interaction vertices.) For our purposes, there are really two kinds of collinear singularity: (a) for $++ \to +$ and $+- \to -$, the number of negative helicity gluons is conserved; (b) for $-- \to -$ and $+- \to +$, the number of negative helicity gluons is reduced by one.

In case (a), the limit as gluons i and $i+1$ become collinear is extracted by merely taking the collinear limit of the MHV vertex to which they are attached. This MHV vertex is a standard Yang-Mills scattering amplitude and has the standard collinear singularities. (Our off-shell continuation of an MHV vertex is easily seen not to modify the collinear singularities for the on-shell particles in that vertex.) Thus, it is manifest that MHV tree diagrams have the correct collinear singularities of type (a).

For singularities of type (b), we need only to be a little more careful. Consider an MHV tree diagram in which gluons i and $i+1$, of helicities $--$ or $+-$, are attached to a vertex with k gluons (some of which may be off-shell), for some $k \geq 3$. If $k \geq 4$, this diagram will not contribute any collinear singularity of type (b), since the "mostly plus" MHV tree amplitudes do not have any $-- \to -$ or $+- \to +$ collinear singularities.

The collinear singularities of type (b) will therefore come entirely from diagrams with $k = 3$, as in figure 5. In the collinear limit, P becomes on-shell, and the spinor λ_P as we have defined it (namely $\lambda_{P\,a} = P_{a\dot{a}}\eta^{\dot{a}}$) becomes a multiple of the spinor arising in the factorization $P_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$. A rescaling of λ_P does not matter. So assuming inductively that the MHV tree diagrams give correctly the $n-1$ gluon tree amplitude (represented by the "blob" in figure 5), the configuration of figure 5 manifestly gives the correct type (b) collinear singularity.

One can see in a similar fashion that our recipe reproduces the correct multiparticle poles in tree amplitudes. Consider an n-particle amplitude and pick some i and j such that the set of particles $i, i+1, \ldots, j$ and the set $j+1, j+2, \ldots, i-1$ each have at least three elements. Let $P = p_i + p_{i+1} + \cdots + p_j$. The multiparticle singularity in this channel has an amplitude that is simply $1/P^2$ times the product of the tree amplitudes in the subchannels. It arises in our formalism from MHV tree diagrams (figure 6) with a single offshell gluon of momentum P connecting the two clusters. The propagator of the off-shell gluon is $1/P^2$, and as $P^2 \to 0$, the spinor λ_P of this gluon, as we have defined it, becomes the standard spinor of an on-shell gluon of momentum P. So assuming inductively that MHV tree graphs describe the lower order amplitudes correctly, they describe the correct multiparticle poles in the n gluon amplitude.

JHEP09(2004)006

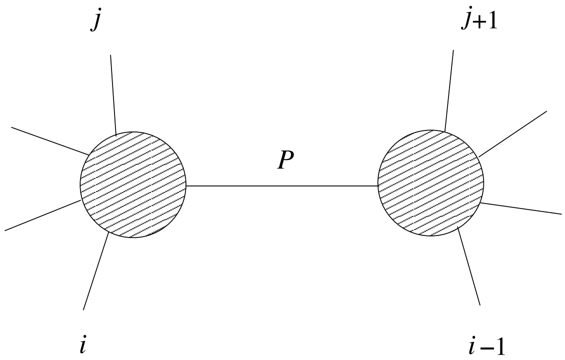


Figure 6: MHV tree diagrams with multiparticle singularities in a particular channel that carries momentum P. The shaded "blobs" represent subamplitudes computed with MHV tree diagrams.

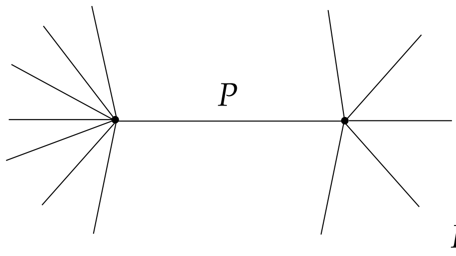


Figure 7: MHV diagrams with two vertices, labeled L and R, connected by a propagator that carries momentum P.

5. Covariance of the amplitudes

Here we will demonstrate that the sum of MHV tree amplitudes is Lorentz covariant. For simplicity, we consider the case of diagrams with only one propagator (and therefore precisely three external gluons of negative helicity), but we do not believe that this restriction is essential. We present the argument here without relation to twistor theory, because the covariance of the sum of MHV trees is of interest irrespective of any connection to twistors. However, the argument was suggested by a nonrigorous twistor analysis that we present in the next section.

Consider as in figure 7 an n-gluon tree diagram with one propagator. The external gluons are divided into two sets L and R of gluons attached to the left or right in the diagram; the internal line carries a momentum $P = \sum_{i\in L} p_i$. We have no natural way to assign spinors λ, $\widetilde{\lambda}$ to the internal line (since in general $P^2 \neq 0$), so instead we introduce an arbitrary λ and $\widetilde{\lambda}$ associated with this line; we will integrate over λ and $\widetilde{\lambda}$ in a manner that will be described.

The gluons attached on the left vertex of figure 7 make up a set L' consisting of L plus the internal gluon, and similarly the gluons on the right make up a set R' consisting of R plus the internal gluon. L' and R' each comes with a natural cyclic order. In an MHV tree diagram, the amplitudes at the left and right vertices are

$$\begin{aligned} g_L(\lambda_i|_{i\in L'}) &= \frac{\langle \lambda_{x_L}, \lambda_{y_L}\rangle^4}{\prod_{i\in L'}\langle \lambda_i, \lambda_{i+1}\rangle} \\ g_R(\lambda_i|_{i\in R'}) &= \frac{\langle \lambda_{x_R}, \lambda_{y_R}\rangle^4}{\prod_{i\in R'}\langle \lambda_i, \lambda_{i+1}\rangle}\,. \end{aligned} \tag{5.1}$$

x_L and y_L are the labels of the negative helicity gluons on the left, and x_R, y_R are the analogous labels on the right. The dependence of $g = g_L g_R$ on λ is extremely simple:

$$g = \frac{\langle \lambda_\sigma, \lambda\rangle^4}{\prod_{\alpha=1}^4 \langle \lambda_\alpha, \lambda\rangle}\widetilde{g}\,, \tag{5.2}$$

JHEP09(2004)006

where $\widetilde{g}$ is independent of λ. Here two poles in the denominator come from g_L and two from g_R; α runs over the four gluons that in the cyclic order are adjacent to the internal line on either the left or the right. σ is the negative chirality gluon on the same side (L or R) on which the internal line carries negative helicity. In particular, g is invariant under scalings of λ.

Now we write down the integral that we will consider:

$$I_\Gamma = \frac{i}{2\pi}\int \langle\lambda, d\lambda\rangle[\widetilde{\lambda}, d\widetilde{\lambda}]\frac{1}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2}g(\lambda;\lambda_i)\,. \tag{5.3}$$

The integration "contour" is described momentarily. We call this integral I_Γ to emphasize the fact that it depends on the choice of a particular MHV tree graph Γ. Since g is invariant under scalings of λ or $\widetilde{\lambda}$ (and in fact is independent of $\widetilde{\lambda}$), the integrand in (5.3) is also invariant under this scaling and makes sense as a meromorphic two-form on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Here the λ^a are homogeneous coordinates on one $\mathbb{CP}^1$, and $\widetilde{\lambda}^{\dot{a}}$ on the second $\mathbb{CP}^1$.

When we actually evaluate the integral, we will take the integration "contour" to be a two-sphere S defined by saying that $\widetilde{\lambda}$ is the complex conjugate of λ. This ensures that the vector $w_{a\dot{a}} = \lambda_a\widetilde{\lambda}_{\dot{a}}$ is real, nonzero, and lightlike. It follows that if P is real and timelike, the denominator $(P_{a\dot{a}}w^{a\dot{a}})^2$ in the definition of I_Γ is everywhere nonzero. The only singularities of the integrand are the simple poles of g, which do not affect the convergence of the integral. The integral over S is hence convergent for timelike P. We use the integral to define I_Γ as an analytic function of P (and the other variables) which can then be continued beyond the real, timelike region. In fact, our evaluation of the integral will give such a continuation.

For the moment, however, we continue algebraically without interpreting $\widetilde{\lambda}$ as the complex conjugate of λ. As in the definition of MHV tree diagrams, we introduce an arbitrary spinor $\eta^{\dot{a}}$ of negative chirality, and we find the identity

$$\frac{[\widetilde{\lambda}, d\widetilde{\lambda}]}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2} = -d\widetilde{\lambda}^{\dot{c}}\frac{\partial}{\partial\widetilde{\lambda}^{\dot{c}}}\left(\frac{[\widetilde{\lambda}, \eta]}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})(P_{b\dot{b}}\lambda^b\eta^{\dot{b}})}\right). \tag{5.4}$$

Since g is independent of $\widetilde{\lambda}$, it trivially follows that likewise

$$\frac{[\widetilde{\lambda}, d\widetilde{\lambda}]\, g(\lambda;\lambda_i)}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2} = -d\widetilde{\lambda}^{\dot{c}}\frac{\partial}{\partial\widetilde{\lambda}^{\dot{c}}}\left(\frac{[\widetilde{\lambda}, \eta]\, g(\lambda;\lambda_i)}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})(P_{b\dot{b}}\lambda^b\eta^{\dot{b}})}\right). \tag{5.5}$$

At this point, we interpret $\widetilde{\lambda}$ as the complex conjugate of λ. If $\lambda^a = (1, z)$, then $\widetilde{\lambda}^{\dot{a}} = (1, \overline{z})$; the integration region S is the complex z plane including a point at infinity. The operator $d\widetilde{\lambda}^{\dot{a}}\partial/\partial\widetilde{\lambda}^{\dot{a}}$ is $d\overline{z}(\partial/\partial\overline{z})$, and if (5.5) were precisely true, it would follow upon integration by parts that I_Γ is identically zero. Actually, once we interpret $\widetilde{\lambda}$ as the complex conjugate of λ, the formula acquires delta function contributions, since

$$\frac{\partial}{\partial\overline{z}}\frac{1}{z-b} = 2\pi\delta(z-b)\,. \tag{5.6}$$

The delta function is normalized so that $\int |dz\, d\overline{z}|\,\delta(z-b) = 1$. This also means that in terms of differential forms, $\int dz \wedge d\overline{z}\,\delta(z-b) = -i = -\int d\overline{z}\wedge dz\,\delta(z-b)$, since if $z = x + iy$

JHEP09(2004)006

with x, y real, then $dz \wedge d\overline{z} = -2idx \wedge dy = -i|dz\, d\overline{z}|$. It is also convenient to write $\overline{\delta}(z-b) = \delta(z-b)d\overline{z}$, and more generally, for any holomorphic function f,

$$\overline{\delta}(f) = \delta(f)d\overline{f}\,. \tag{5.7}$$

(Thus $\overline{\delta}(f)$ is a $\overline{\partial}$-closed $(0,1)$-form, a property that we will use in section 6.) So

$$\int dz\, \overline{\delta}(z-b) = -i\,. \tag{5.8}$$

We can write (5.6) in a more covariant form:

$$d\widetilde{\lambda}^{\dot{c}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{c}}} \frac{1}{\langle \zeta, \lambda \rangle} = 2\pi \overline{\delta}(\langle \zeta, \lambda \rangle)\,, \tag{5.9}$$

again assuming $\widetilde{\lambda} = \overline{\lambda}$. The idea here is that in coordinates with $\lambda^a = (1,z)$, $\widetilde{\lambda}^{\dot{a}} = (1,\overline{z})$, $\zeta^a = (1,b)$, (5.9) reduces to (5.6). If $\lambda^a = (1,z)$, then $\langle \lambda, d\lambda \rangle = dz$, so a more covariant version of (5.8) is the statement that if $B(\lambda)$ is any function that is homogeneous of degree -1, then

$$\int \langle \lambda, d\lambda \rangle \overline{\delta}(\langle \zeta, \lambda \rangle) B(\lambda) = -iB(\zeta)\,. \tag{5.10}$$

In evaluating (5.5) more precisely to include such delta functions, we need not be concerned about singularities from zeroes of $P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}}$, since as we have discussed, this function has no zeroes in the integration region. However, we get a contribution that we will call $I_{\Gamma,\eta}$ from the pole at

$$\lambda_a = P_{a\dot{a}}\eta^{\dot{a}} \tag{5.11}$$

that comes from the vanishing of the factor $P_{a\dot{a}}\lambda^a\eta^{\dot{a}}$ in the denominator. And we get four contributions that we will call $I_{\Gamma,\alpha}$ from the poles at $\lambda = \lambda_\alpha$ which are visible in the formula (5.2) for g. The condition (5.11) should be familiar; it was used in section 2 to make an off-shell continuation of the MHV amplitudes.

So we can schematically write

$$I_\Gamma = I_{\Gamma,\eta} + \sum_\alpha I_{\Gamma,\alpha}\,. \tag{5.12}$$

To evaluate $I_{\Gamma,\eta}$, and $I_{\Gamma,\alpha}$, we evaluate (5.5) more precisely, including the delta functions that should be included when $\widetilde{\lambda}$ is understood as the complex conjugate of λ. We have

$$\begin{aligned}\frac{[\widetilde{\lambda}, d\widetilde{\lambda}]\, g(\lambda;\lambda_i)}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2} &= -d\widetilde{\lambda}^{\dot{c}}\frac{\partial}{\partial\widetilde{\lambda}^{\dot{c}}}\left(\frac{[\widetilde{\lambda},\eta]\, g(\lambda;\lambda_i)}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})(P_{b\dot{b}}\lambda^b\eta^{\dot{b}})}\right) + \\ &+\frac{2\pi[\widetilde{\lambda},\eta]}{P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}}}\left(-\overline{\delta}(P_{b\dot{b}}\lambda^b\eta^{\dot{b}})g + \frac{1}{P_{b\dot{b}}\lambda^b\eta^{\dot{b}}}\sum_{\alpha=1}^{4}\overline{\delta}(\langle\lambda_\alpha,\lambda\rangle)\frac{\langle\lambda_\sigma,\lambda_\alpha\rangle^4}{\prod_{\beta\neq\alpha}\langle\lambda_\beta,\lambda_\alpha\rangle}\widetilde{g}\right).\end{aligned} \tag{5.13}$$

We can now evaluate $I_{\Gamma,\eta}$, which is the contribution of the delta function that is supported at $\lambda_a = P_{a\dot{a}}\eta^{\dot{a}}$. At $\lambda_a = P_{a\dot{a}}\eta^{\dot{a}}$, we have $[\widetilde{\lambda},\eta]/P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}} = -1/(\frac{1}{2}P_{a\dot{a}}P^{a\dot{a}}) = -1/P^2$. So

$$I_{\Gamma,\eta} = \frac{1}{P^2}g(\lambda_P;\lambda_i)\,, \tag{5.14}$$

JHEP09(2004)006

where as in section 2, $\lambda_{P\,a} = P_{a\dot a}\eta^{\dot a}$. In other words, $I_{\Gamma,\eta}$ is simply the amplitude, as defined in section 2, for the MHV tree graph Γ. Similarly,

$$I_{\Gamma,\alpha} = \frac{2\pi[\widetilde{\lambda}_\alpha,\eta]}{(P_{a\dot a}\lambda_\alpha^a\widetilde{\lambda}_\alpha^{\dot a})(P_{b\dot b}\lambda_\alpha^b\eta^{\dot b})}\frac{\langle\lambda_\sigma,\lambda_\alpha\rangle^4}{\prod_{\beta\neq\alpha}\langle\lambda_\beta,\lambda_\alpha\rangle}\widetilde{g} = \frac{2\pi[\widetilde{\lambda},\eta]}{(P_{a\dot a}\lambda_\alpha^a\widetilde{\lambda}_\alpha^{\dot a})(P_{b\dot b}\lambda_\alpha^b\eta^{\dot b})}\mathrm{Res}_{\lambda=\lambda_\alpha}\, g(\lambda;\lambda_i)\,. \tag{5.15}$$

Upon summing over all tree graphs with the given set of external gluons, we have

$$\sum_\Gamma I_\Gamma = \sum_\Gamma I_{\Gamma,\eta} + \sum_\alpha I_{\Gamma,\alpha}\,. \tag{5.16}$$

We will see shortly that

$$\sum_\Gamma I_{\Gamma,\alpha} = 0 \tag{5.17}$$

for all α. Given this, we have

$$\sum_\Gamma I_\Gamma = \sum_\Gamma I_{\Gamma,\eta}\,. \tag{5.18}$$

Since the left hand side is Lorentz covariant (a statement that we explain more fully below), it follows as we have promised that the sum of MHV tree amplitudes is covariant.

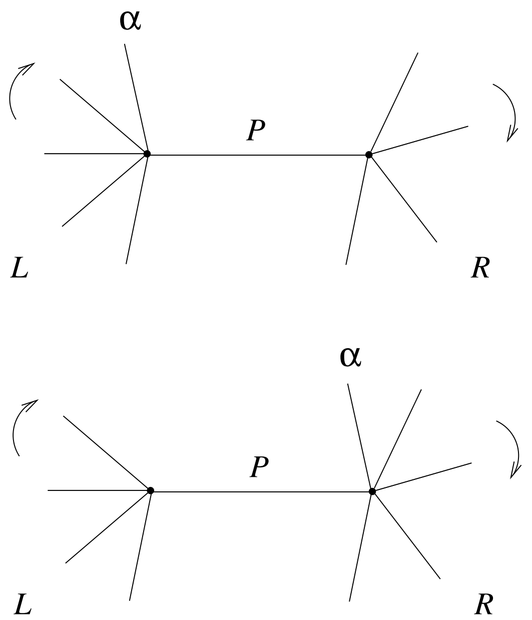


Figure 8: The graphs contributing a pole at $\lambda = \lambda_\alpha$. Each vertex has a natural cyclic order, which we take to be counterclockwise, as indicated by the arrows. In one graph, α is on the left, just ahead of the internal line, and in the other graph, it is on the right, just after it. The reversed order reverses the sign of the residue of the pole.

Now we will verify (5.17). We consider two graphs Γ_1 and Γ_2 — selected as in figure 8 — for which the function g has a pole at $\lambda = \lambda_\alpha$. They differ by whether the gluon α is in L, just before the internal gluon (in the cyclic order), or in R, just after it. Because of this difference in ordering, when we evaluate $g = g_L g_R$ using (5.1), one g function contains a factor $1/\langle\lambda,\lambda_\alpha\rangle$ while the other contains a factor $1/\langle\lambda_\alpha,\lambda\rangle$. The other factors in the two g functions, which we will call g_1 and g_2, become equal when we set $\lambda = \lambda_\alpha$. So $\mathrm{Res}_{\lambda=\lambda_\alpha}\, g_1 = -\mathrm{Res}_{\lambda=\lambda_\alpha}\, g_2$. The other factor in (5.15) that we must consider in comparing $I_{\Gamma_1,\alpha}$ and $I_{\Gamma_2,\alpha}$ is $X = 1/(P_{a\dot a}\lambda_\alpha^a\widetilde{\lambda}_\alpha^{\dot a})(P_{b\dot b}\lambda_\alpha^b\eta^{\dot b})$. The two graphs have different P's, but as the P's differ by $P_{a\dot a} \to P_{a\dot a} + \lambda_{\alpha\,a}\widetilde{\lambda}_{\alpha\,\dot a}$, they have the same value of X. So finally, the two graphs give equal and opposite poles at $\lambda = \lambda_\alpha$. All poles at $\lambda = \lambda_\alpha$ are canceled in this way among pairs of graphs.

5.1 A subtle detail

There is actually one further subtlety in this argument (which some readers may wish to omit). Suppose that on the left of the first diagram in figure 8 there are only two external gluons — one labeled α and one labeled, say, β. The evaluation of the diagram as above yields a pole at $\lambda = \lambda_\alpha$ that must be canceled by a similar pole when α has moved to the

JHEP09(2004)006

right. In that contribution, only one gluon, namely β, remains on the left (figure 9). We therefore have to allow contributions in the present analysis in which only two gluons (one of them off-shell) are attached to the vertex on the left. This presents a riddle, since the MHV tree diagrams have no such divalent vertices.

Let us see examine this more closely. In figure 9, both β and the internal gluon joining to L have negative helicity (since they are the only candidates for the two negative helicity gluons on L). Hence in (5.2), σ and two of the α's are both equal to β, so g becomes

$$g = \langle \lambda_\beta, \lambda \rangle^2 \prod_\nu \frac{1}{\langle \lambda_\nu, \lambda \rangle} \widetilde{g}\,, \tag{5.19}$$

where ν runs over the two neighbors of the internal gluon in R. In particular, there is no pole at $\lambda = \lambda_\beta$ (and so no need to cancel its residue by introducing a contribution with only one gluon attached to L).

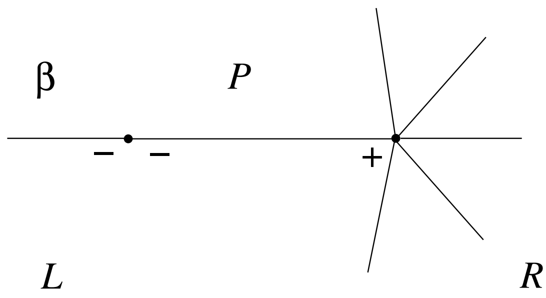


Figure 9: A diagram with a divalent vertex on the left. The two gluons entering the vertex both have negative helicity. The external gluon is labeled β, and its momentum p_β also equals the momentum P of the internal gluon.

Since $P = p_\beta$, we have $P_{a\dot a} = \lambda_{\beta\,a} \widetilde{\lambda}_{\beta\,\dot a}$. The integral representation of I_Γ becomes

$$I_\Gamma = \frac{i}{2\pi} \int \langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}] \frac{1}{[\widetilde{\lambda}, \widetilde{\lambda}_\beta]^2} \prod_\nu \frac{1}{\langle \lambda_\nu, \lambda \rangle} \widetilde{g}\,, \tag{5.20}$$

where a factor of $\langle \lambda, \lambda_\beta \rangle^2$ in the denominator has canceled such a factor in the numerator of (5.19). This cancellation ensures that the integral for I_Γ is convergent (if we integrate symmetrically near $\widetilde{\lambda} = \widetilde{\lambda}_\beta$ where the denominator has its strongest singularity) even though P is lightlike.

We can evaluate the integral by a sum of residues. First consider the contribution $I_{\Gamma,\nu}$ from the pole at $\lambda = \lambda_\nu$ (for either of the two possible values of ν). By (5.15), it is

$$I_{\Gamma,\nu} = \frac{2\pi [\widetilde{\lambda}_\nu, \eta]}{(P_{a\dot a} \lambda_\nu^a \widetilde{\lambda}_\nu^{\dot a})(P_{b\dot b} \lambda_\nu^b \eta^{\dot b})} \mathrm{Res}_{\lambda=\lambda_\nu} g(\lambda; \lambda_i)\,. \tag{5.21}$$

With only one gluon on L, we have $P = p_\beta$. So $P_{b\dot b} = \lambda_{\beta\,b} \widetilde{\lambda}_{\beta\,\dot b}$, whence

$$I_{\Gamma,\nu} = \frac{2\pi [\widetilde{\lambda}_\nu, \eta]}{\langle \lambda_\beta, \lambda_\nu \rangle^2 [\widetilde{\lambda}_\beta, \widetilde{\lambda}_\nu][\widetilde{\lambda}_\beta, \eta]} \mathrm{Res}_{\lambda=\lambda_\nu} g(\lambda; \lambda_i)\,. \tag{5.22}$$

From (5.19), if we write ν_i, $i = 1, 2$ for the two possible values of ν, this gives

$$I_{\Gamma,\nu_i} = \frac{2\pi [\widetilde{\lambda}_{\nu_i}, \eta]}{[\widetilde{\lambda}_\beta, \widetilde{\lambda}_{\nu_i}][\widetilde{\lambda}_\beta, \eta]} \frac{\widetilde{g}}{\langle \lambda_{\nu_{i'}}, \lambda_{\nu_i} \rangle}\,, \tag{5.23}$$

where $\nu_{i'} \neq \nu_i$. From this it follows (using the Schouten identity of footnote 2 to combine the terms) that $\sum_{i=1,2} I_{\Gamma,\nu_i}$ is independent of η. Hence, unlike the cases with more than two gluons attached to L, we do not have to add an additional contribution from a pole at $\lambda_a = P_{a\dot a} \eta^{\dot a}$ to cancel the η-dependence.

JHEP09(2004)006

We do not want such a contribution, since, with the vertex on the left of figure 9 being divalent, it does not correspond to anything in the MHV tree diagrams of sections 2 and 3. In more general cases with a k-valent vertex of $k \geq 3$, the contribution that we called $I_{\Gamma,\eta}$ arises from the singularity of

$$\frac{2\pi[\widetilde{\lambda},\eta]g(\lambda;\lambda_i)}{(P_{b\dot{b}}\lambda^b\widetilde{\lambda}^{\dot{b}})(P_{a\dot{a}}\lambda^a\eta^{\dot{a}})} \tag{5.24}$$

at $P_{a\dot{a}}\lambda^a\eta^{\dot{a}} = 0$. With $P_{a\dot{a}} = \lambda_{\beta\,a}\widetilde{\lambda}_{\beta\,\dot{a}}$, this singularity would be at $\langle\lambda_\beta,\lambda\rangle = 0$, but in (5.24), there is no singularity there, because g is divisible by $\langle\lambda_\beta,\lambda\rangle^2$. Thus, configurations with a divalent vertex have a nonvanishing $I_{\Gamma,\alpha}$ and participate in the associated cancellation, but have vanishing $I_{\Gamma,\eta}$ and do not contribute to the MHV tree diagrams.

5.2 Covariance of the amplitude

Finally, the assertion that $I = \sum_\Gamma I_\Gamma$ is Lorentz covariant needs some elaboration:

1. The integral representation (5.3) appears to show that I_Γ is holomorphic in the λ_i and in the $\widetilde{\lambda}_i$ (the latter enter only via P). Though the holomorphy in $\widetilde{\lambda}_i$ is valid, the holomorphy in λ_i fails because of the poles: the $\overline{\partial}$ operator of λ_α, namely $d\overline{\lambda}^a_\alpha\,\partial/\partial\overline{\lambda}^a_\alpha$, in acting on the integrand of I_Γ, produces a delta function at $\lambda = \lambda_\alpha$. When we write $I_\Gamma = I_{\Gamma,\eta} + \sum_\alpha I_{\Gamma,\alpha}$, the first term $I_{\Gamma,\eta}$ is holomorphic in the λ_α, but the $I_{\Gamma,\alpha}$ are not.

2. The integral (5.3) defining I_Γ formally has $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ symmetry, where one SL(2) acts on spinor indices a, b and the other on spinor indices $\dot{a}, \dot{b}$. Thus, one SL(2) acts on λ, λ_i, and the other on $\widetilde{\lambda}, \widetilde{\lambda}_i$. $\mathrm{SL}(2) \times \mathrm{SL}(2)$ is a double cover of the complexified Lorentz group.

3. The choice of integration contour S with $\widetilde{\lambda} = \overline{\lambda}$ breaks $\mathrm{SL}(2) \times \mathrm{SL}(2)$ down to the diagonal SL(2), which is a double cover of the real Lorentz group SO(3,1). Were there no poles, a contour deformation argument would show that the integral possesses the full $\mathrm{SL}(2)\times\mathrm{SL}(2)$ symmetry, even though the contour does not. Because of the poles, the full $\mathrm{SL}(2) \times \mathrm{SL}(2)$ invariance is not restored upon doing the integral and I_Γ is only invariant under the diagonal SL(2).

4. After summing over Γ, the $I_{\Gamma,\alpha}$ cancel, as we argued above, and hence holomorphy in the λ_i is restored.

5. The sum $I = \sum_\Gamma I_\Gamma = \sum_\Gamma I_{\Gamma,\eta}$ is accordingly holomorphic in the λ_i and $\widetilde{\lambda}_i$. The real Lorentz group, or rather its double cover SL(2), acts holomorphically on these variables leaving I invariant, and hence I is automatically invariant under the complexification of this group, which is the full $\mathrm{SL}(2) \times \mathrm{SL}(2)$.

6. Heuristic analysis of disconnected twistor diagrams

Here we will make a nonrigorous analysis of the disconnected twistor diagrams that contribute to the amplitudes studied in the last section. Interpreting the interaction vertices

JHEP09(2004)006

in the Feynman diagram of figure 7 as degree one instantons in twistor space, and the line connecting the vertices as a twistor propagator, we will explain what manipulations applied to this twistor configuration give the integral studied in the last section.

We are going to use somewhat different twistor space wavefunctions than those used in [6]. We take our particles to have definite momenta $p_i^{a\dot{a}} = \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}$ in Minkowski space. The corresponding twistor space wavefunction is[3]

$$\overline{\delta}(\langle \lambda, \lambda_i \rangle) \exp(i[\mu, \widetilde{\lambda}_i])\,. \tag{6.1}$$

The idea here is that this wavefunction represents a particle of definite λ because the wavefunction has delta function support at $\lambda = \lambda_i$, and it has definite $\widetilde{\lambda}$ because of the plane wave dependence on μ. Choosing twistor space wavefunctions that represent momentum space eigenstates in Minkowski space means that the twistor computation can be compared directly to the standard momentum space scattering amplitudes, without needing to perform an additional Fourier transform. It turns out that this also simplifies the computations. (The same simplification was achieved in [9] by performing a Fourier transform prior to evaluating the twistor scattering amplitude.)

The effective action for fields in twistor space is the integral of a Chern-Simons $(0,3)$-form. The kinetic operator for these fields is the $\overline{\partial}$ operator. The propagator is a $(0,2)$-form on $\mathbb{CP}^3 \times \mathbb{CP}^3$ that we write as $G(\lambda', \mu'; \lambda, \mu)$, where (λ, μ) are homogeneous coordinates for one point in $\mathbb{CP}^3$ and (λ', μ') for the other. The part of G that is a $(0,1)$-form on each copy of $\mathbb{CP}^3$ is the propagator for the physical fields, while as in quantization of real Chern-Simons gauge theory [10], the terms in G that are $(0,2)$-forms on one $\mathbb{CP}^3$ and $(0,0)$-forms on the other describe propagation of ghosts. We write the equation that should be obeyed by G in coordinates with $\lambda^1 = \lambda'^1 = 1$:[4]

$$\overline{\partial} G = \frac{1}{2\pi} \overline{\delta}\left(\lambda'^2 - \lambda^2\right) \overline{\delta}\left(\mu'^{\dot{1}} - \mu^{\dot{1}}\right) \overline{\delta}\left(\mu'^{\dot{2}} - \mu^{\dot{2}}\right). \tag{6.2}$$

We can therefore take the propagator to be

$$G = \frac{1}{(2\pi)^2} \overline{\delta}\left(\lambda'^2 - \lambda^2\right) \overline{\delta}\left(\mu'^{\dot{1}} - \mu^{\dot{1}}\right) \frac{1}{\mu'^{\dot{2}} - \mu^{\dot{2}}}\,. \tag{6.3}$$

This choice of G amounts to a choice of gauge.

Now consider the exchange of a twistor field between copies of $\mathbb{CP}^1$ that represent instantons C and C' of degree one. As in figure 10, the external gluons are attached to C and C'. C is described by the equation

$$\mu^{\dot{a}} = x^{a\dot{a}} \lambda_a\,, \tag{6.4}$$

[3]The twistor space wavefunction is supposed to be a $\overline{\partial}$-closed $(0,1)$-form with values in a line bundle that depends on the helicity. We have a $\overline{\partial}$-closed $(0,1)$-form here because $\overline{\delta}(f)$, for any holomorphic function f, is such a form. Since the line bundles in question are naturally trivial when restricted to $\lambda = \lambda_i$, we can (at the informal level of the present discussion) write the wavefunctions without being very precise in describing the line bundle.

[4]The prefactor $1/2\pi$ depends on the proper normalization of the Chern-Simons $(0,3)$-form action in twistor space. We are making a guess based on the analogous normalization for real Chern-Simons theory at level one and will not try to prove that this is the correct normalization of the propagator.

JHEP09(2004)006

and C' by the equation

$$\mu'^{\dot a} = x'^{a\dot a}\lambda'_a\,. \tag{6.5}$$

We also set $y_{a\dot a} = x'_{a\dot a} - x_{a\dot a}$. C' and C will correspond respectively to the vertices on the left and right of figure 7.

The exponential factors in (6.1) give an important dependence on x and x'. Taking the product of the exponentials for all of the external particles, we get

$$\prod_{i\in L}\exp(ix'_{a\dot a}p_i^{a\dot a})\prod_{j\in R}\exp\left(ix_{b\dot b}p_j^{b\dot b}\right). \tag{6.6}$$

We can also write this expression as

$$\exp(iy_{a\dot a}P^{a\dot a})\prod_{i}\exp\left(ix_{b\dot b}p_i^{b\dot b}\right), \tag{6.7}$$

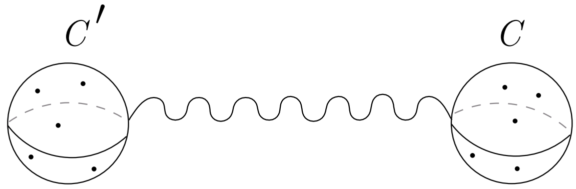


Figure 10: Twistor diagrams corresponding to MHV tree diagrams that were considered in section 5. There are two disconnected instantons, labeled C and C', to which gluons are attached; they are connected by a twistor space propagator.

where as in section 5, $P = \sum_{i\in L} p_i$, and in the second factor all external particles are included. The integral over x will give a delta function of energy-momentum conservation; the y-dependent factor in (6.7) will also play an important role.

We will take the measure for integrating over x and y to be $d^4x^{a\dot a}\,d^4y^{b\dot b}$, where, for example, $d^4y^{b\dot b} = dy^{1\dot 1}dy^{2\dot 2}dy^{2\dot 1}dy^{1\dot 2}$.

With our choice of gauge, in coordinates with $\lambda^1 = \lambda'^1 = 1$, the twistor propagator G is supported on pairs of points that obey $\lambda'^2 = \lambda^2$. We can more invariantly say simply that $\lambda'^a = \lambda^a$ (without specializing to coordinates with $\lambda^1 = 1$). In addition, as $\mu'^{\dot a} - \mu^{\dot a} = y^{a\dot a}\lambda_a$, the condition that the propagator is exchanged between points with $\mu'^{\dot 1} = \mu^{\dot 1}$ means that $y^{a\dot 1}\lambda_a = 0$, or in other words that

$$\lambda^a = y^{a\dot 1} \tag{6.8}$$

up to an irrelevant scaling. The propagator contains a factor $1/(\mu'^{\dot 2} - \mu^{\dot 2}) = 1/y^{a\dot 2}\lambda_a = 1/y^{a\dot 2}y_a{}^{\dot 1}$. But $y^{a\dot 2}y_a{}^{\dot 1} = \frac{1}{2}(y^{a\dot 2}y_a{}^{\dot 1} - y^{a\dot 1}y_a{}^{\dot 2}) = -y^{a\dot a}y_{a\dot a}/2$.

So finally the integral representing the contribution $\widetilde{I}_\Gamma$ to the scattering amplitude from the instanton configuration considered in figure 10 is

$$\widetilde{I}_\Gamma = -\frac{1}{2\pi^2}\int\frac{d^4y^{b\dot b}}{y^{a\dot a}y_{a\dot a}}\exp(iy_{c\dot c}P^{c\dot c})g(\lambda;\lambda_i)\,, \tag{6.9}$$

where $\lambda^a = y^{a\dot 1}$, while λ_i^a are the spinors associated with external gluons. The function $g(\lambda;\lambda_i)$ arises from computing the correlation function of gluon vertex operators on C and C' (and integrating over fermionic moduli) as explained in section 4.7 of [6]. It is the same function that entered in section 5. (The factor $\exp(iy_{a\dot a}P^{a\dot a})$ was absent in analogous formulas in [6] because different twistor space wavefunctions were used.) Most of the ingredients in (6.9) are Lorentz-covariant; Lorentz covariance is violated only because the function $g(\lambda;\lambda_i)$ is evaluated at $\lambda^a = y^{a\dot 1}$, clearly a noncovariant condition.

We now have to decide how to interpret the integral in (6.9). The integrand is a holomorphic function of y and the integral is a complex contour integral of some sort. We

JHEP09(2004)006

most definitely do not know any systematic theory of how to pick the contours in topological string theory in twistor space. Here we will simply describe a recipe for interpreting this integral that was found in an attempt to match with our results about MHV tree diagrams.

We assume, first of all, that one of the y integrals should be performed via a contour integral around the pole at $y^2 = 0$, and thus gives $2\pi i$ times the residue of that pole. The integral thus becomes an integral on the quadric Q defined by $y^2 = 0$. We write this schematically

$$\widetilde{I}_\Gamma = -\frac{i}{\pi}\int_Q \mathrm{Res}_{y^2=0}\frac{d^4y^{b\dot{b}}}{y_{c\dot{c}}y^{c\dot{c}}}\,\exp(iy_{a\dot{a}}P^{a\dot{a}})g(\lambda,\lambda_i)\,. \tag{6.10}$$

(We will compute this residue momentarily.) Once this is done, our formula becomes Lorentz-invariant. Indeed, at $y^2 = 0$, we can factor $y^{a\dot{a}}$ as $\lambda^a\widetilde{\lambda}^{\dot{a}}$, where one way to determine λ is to say that up to an irrelevant scaling, $\lambda^a = y^{a\dot{1}}$. In fact, the formula $y^{a\dot{a}} = \lambda^a\widetilde{\lambda}^{\dot{a}}$ implies $\lambda^a = y^{a\dot{1}}/\widetilde{\lambda}^{\dot{1}}$.

We actually want to decompose $y^{a\dot{a}}$ a little differently. We write

$$y^{a\dot{a}} = t\lambda^a\widetilde{\lambda}^{\dot{a}}\,, \tag{6.11}$$

where the λ^a are homogeneous coordinates for one copy of $\mathbb{CP}^1$, $\widetilde{\lambda}^{\dot{a}}$ are homogeneous coordinates for a second copy of $\mathbb{CP}^1$, and t scales with weight -1 under scaling of either λ or $\widetilde{\lambda}$. The scaling of t has been selected to ensure that y is invariant. The measure on the quadric is determined by the symmetries to be

$$\mathrm{Res}_{y^2=0}\frac{d^4y}{y^2} = f\,t\,dt\,\langle\lambda,d\lambda\rangle[\widetilde{\lambda},d\widetilde{\lambda}]\,, \tag{6.12}$$

for some constant f (which we will soon find to equal $1/2$). The dependence on λ and $\widetilde{\lambda}$ is determined from $\mathrm{SL}(2)\times\mathrm{SL}(2)$ invariance; the power of t can be fixed by requiring that the measure is invariant under scaling of λ or $\widetilde{\lambda}$.

To compute f, we simply compare the two measures at a convenient point P. The differential form $d^4y^{a\dot{a}}/y^{b\dot{b}}y_{b\dot{b}} = dy^{1\dot{1}}dy^{2\dot{2}}dy^{2\dot{1}}dy^{1\dot{2}}/2(y^{1\dot{1}}y^{2\dot{2}} - y^{1\dot{2}}y^{2\dot{1}})$ has a pole at $y^{2\dot{2}} = y^{1\dot{2}}y^{2\dot{1}}/y^{1\dot{1}}$ whose residue is the volume form $\Phi = dy^{1\dot{1}}dy^{2\dot{1}}dy^{1\dot{2}}/2y^{1\dot{1}}$ on Q. The point P at which the only nonzero component of y is $y^{1\dot{1}} = 1$ corresponds in the other variables to $t = 1$, $\lambda^a = (1,0)$, $\widetilde{\lambda}^{\dot{a}} = (1,0)$. Expanding around this point, we take $t = 1+\epsilon$, $\lambda^a = (1,\beta)$, $\widetilde{\lambda}^{\dot{a}} = (1,\gamma)$, whence to first order $y^{1\dot{1}} = 1+\epsilon$, $y^{2\dot{1}} = \beta$, $y^{1\dot{2}} = \gamma$. So at P, $\Phi = d\epsilon\,d\beta\,d\gamma/2$. On the other hand, $dt = d\epsilon$, $\langle\lambda,d\lambda\rangle = d\beta$, and $d\widetilde{\lambda} = d\gamma$. So $t\,dt\langle\lambda,d\lambda\rangle[\widetilde{\lambda},d\widetilde{\lambda}] = d\epsilon\,d\beta\,d\gamma$. Comparing these formulas, we find that $f = 1/2$.

Our integral therefore becomes

$$\widetilde{I}_\Gamma = -\frac{i}{2\pi}\int t\,dt\langle\lambda,d\lambda\rangle[\widetilde{\lambda},d\widetilde{\lambda}]\exp(it\lambda_a\widetilde{\lambda}_{\dot{a}}P^{a\dot{a}})\;g(\lambda;\lambda_i)\,. \tag{6.13}$$

Again, the proper interpretation of this integral is unclear. Trying to get an answer that makes some sense, we interpret the t integral as an integral from 0 to ∞, using

$$\int_0^\infty t\,dt\,\exp(i\gamma t) = -\frac{1}{\gamma^2}\,. \tag{6.14}$$

JHEP09(2004)006

So

$$\widetilde{I}_\Gamma = \frac{i}{2\pi}\int \langle\lambda, d\lambda\rangle[\widetilde{\lambda}, d\widetilde{\lambda}]\frac{1}{(P_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})^2}g(\lambda;\lambda_i)\,. \tag{6.15}$$

Thus we have motivated the integral that we took as our starting point in section 5.[5]

Integrating over t from 0 to ∞ seems rather unpalatable in the context of complex contour integrals, since contours are normally closed cycles or cycles that run off to infinity (rather than terminating at $t = 0$). The procedure that we have followed seems somewhat more plausible in conjunction with the choice we made in section 5 of setting $\widetilde{\lambda} = \overline{\lambda}$. With $y^{a\dot{a}} = t\lambda^a\widetilde{\lambda}^{\dot{a}}$, the combined operation of setting $\widetilde{\lambda} = \overline{\lambda}$ and taking t to be real and positive amounts to integrating over the future light cone in real Minkowski spacetime; this seems like a more or less respectable integration cycle, albeit singular at the origin and noncompact.

The proof of $\mathrm{SL}(2,\mathbb{C})$ invariance at the end of section 5 shows that integration over the past light cone would give the same result. In fact, unlike the real light cone, which has a future and a past, the complexified light cone is connected. We have shown that our amplitudes, after summing over graphs, are invariant under the complexified Lorentz group. This group can be used to rotate the future real light cone to the past real light cone.

What shall we make of our result? If our procedure for calculating the integral is correct, then the tree level Yang-Mills scattering amplitudes appear to come from totally disconnected instanton configurations in twistor space. On the other hand, there appears to be convincing evidence [9] that they can be computed from connected instantons alone. Are there really two distinct ways to compute the same amplitudes from twistor space? For more general amplitudes, are there more than two ways, allowing for instantons of higher degree that are neither connected nor completely disconnected? Or is there a fault in the way the integrals have been evaluated? Certainly, we cannot claim a firm justification for the way that we have evaluated the integral. So in our computation, there are ample possibilities to suppose that a more complete and rigorous evaluation of the scattering amplitude might require including additional contributions.

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JHEP09(2004)006

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JHEP09(2004)006